

INTERVAL ORDERS AND REVERSE MATHEMATICS

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ABSTRACT. We study the reverse mathematics of interval orders. We establish the logical strength of the implications between various definitions of the notion of interval order. We also consider the strength of different versions of the characterization theorem for interval orders: a partial order is an interval order if and only if it does not contain $\mathbf{2} \oplus \mathbf{2}$. We also study proper interval orders and their characterization theorem: a partial order is a proper interval order if and only if it contains neither $\mathbf{2} \oplus \mathbf{2}$ nor $\mathbf{3} \oplus \mathbf{1}$.

Interval orders are a particular kind of partial orders which occur quite naturally in many different areas and have been widely studied. A partial order $\mathbf{P} = (P, \leq_P)$ is an *interval order* if the elements of P can be mapped to nonempty intervals of a linear order \mathbf{L} so that $p <_P q$ holds iff every element of the interval associated to p precedes every element of the interval associated to q . The linear order \mathbf{L} and the map from P to intervals are called an *interval representation of \mathbf{P}* . The basic reference on interval orders is Fishburn's monograph [9].

The name "interval order" was introduced by Fishburn ([8]), although the notion was already studied much earlier by Norbert Wiener ([23]), who used the terminology "relation of complete sequence". Interval orders model many phenomena occurring in the applied sciences: [9, §2.1] include examples such as chronological dating in archaeology and paleontology, scheduling of manufacturing processes, and psychophysical perception of sounds. Notice that if \mathbf{P} is a countable interval order then we can assume that \mathbf{L} is the rational or (as usual in applications) the real line (a *real representation*, in the terminology of [9]).

Most recent research on interval orders (see e.g. the survey [22] and chapter 8 of [18]) focuses on finite partial orders, while in this paper we consider mostly infinite ones (although a careful analysis of the finite case is instrumental in obtaining results in the infinite case). A recent result about infinite interval orders shows that every interval order which is a well quasi-order is a better quasi-order ([15]).

The basic characterization for interval orders is given by the following theorem proved independently by Fishburn ([8]) and Mirkin ([13]):

Characterization Theorem 1. *A partial order is an interval order if and only if it does not contain $\mathbf{2} \oplus \mathbf{2}$.*

Here " \mathbf{P} does not contain $\mathbf{2} \oplus \mathbf{2}$ " means that for no $P' \subseteq P$ the restriction of \leq_P to P' is the partial order with Hasse diagram $\begin{smallmatrix} \bullet & \bullet \\ | & | \\ \bullet & \bullet \end{smallmatrix}$. It is easy to see that \mathbf{P} does not contain $\mathbf{2} \oplus \mathbf{2}$ if and only if

$$\forall p_0, q_0, p_1, q_1 \in P (p_0 \leq_P q_0 \wedge p_1 \leq_P q_1 \implies p_0 \leq_P q_1 \vee p_1 \leq_P q_0).$$

Two natural ways of strengthening the notion of interval order lead to the definitions of unit interval order and proper interval order.

An interval order with a real representation such that all intervals have the same positive length (which can be assumed to be 1) is called a *unit interval order*.

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If an interval order \mathbf{P} has an interval representation such that an interval associated to an element of P is never a proper subset of another such interval, then we say that \mathbf{P} is a *proper interval order*. An interval representation with the above property is called a *proper interval representation*.

It is immediate that every unit interval order is a proper interval order. If the partial order is finite then the reverse implications is also true ([16], see [2] for a short proof). On the other hand, there exist infinite proper interval orders which are not unit interval orders: a simple example is provided by the ordinal $\omega + 1$. Notice however that the fact that $\omega + 1$ is not a unit interval order has more to do with the real line (which in this context appears to be “too short”) than with structural properties of the partial order. Therefore when dealing with infinite partial orders the notion of proper interval order appears to be more natural, as witnessed also by the following characterization theorem.

Characterization Theorem 2. *A partial order is a proper interval order if and only if it contains neither $\mathbf{2} \oplus \mathbf{2}$ nor $\mathbf{3} \oplus \mathbf{1}$.*

“ \mathbf{P} does not contain $\mathbf{3} \oplus \mathbf{1}$ ” means that for no $P' \subseteq P$ the restriction of \leq_P to P' is the partial order with Hasse diagram $\begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array}$. It is easy to see that \mathbf{P} does not contain $\mathbf{3} \oplus \mathbf{1}$ if and only if

$$\forall p_0, p_1, p_2, q \in P (p_0 <_P p_1 <_P p_2 \implies p_0 \leq_P q \vee q \leq_P p_2).$$

Characterization Theorem 2 is usually known as the Scott-Suppes Theorem. Scott and Suppes ([19]) proved the theorem in the finite case for unit interval orders (see [1] for a simple proof in this setting). Fishburn’s monograph includes a proof of this theorem with no restrictions on cardinality ([9, Theorem 2.7]).

In this paper we study interval orders and proper interval orders from the viewpoint of *reverse mathematics*. The basic reference for reverse mathematics is Simpson’s book [20], which contains all background material needed for this paper (and much more). A sample of recent research in the area is contained in [21].

In reverse mathematics, one formalizes theorems of ordinary mathematics and attempts to discover the set theoretic axioms required to prove these theorems. This project is usually carried out in the context of subsystems of second order arithmetic, taking RCA_0 as the base system. RCA_0 is the subsystem obtained from full second order arithmetic by restricting the comprehension scheme to Δ_1^0 formulas and adding a formula induction scheme for Σ_1^0 formulas. In this paper, we will be concerned only with RCA_0 and its fairly weak extension known as WKL_0 (WKL_0 is strictly weaker than the subsystem ACA_0 obtained by extending the comprehension scheme in RCA_0 to all arithmetic formulas). WKL_0 is obtained by adjoining Weak König’s Lemma (i.e. König’s Lemma for trees of sequences of 0’s and 1’s) to RCA_0 .

Many results about partial and linear orders have been studied from the viewpoint of reverse mathematics: recent papers include [6, 5, 4, 3, 10, 11, 12, 14]. Moreover, [17, §3] includes a couple of results about interval graphs, which are strictly connected to interval orders.

1. OVERVIEW OF RESULTS AND PLAN OF THE PAPER

The first step in the study of a new topic in the context of reverse mathematics is finding appropriate formalizations of the relevant notions. Often, this requires making choices between classically equivalent definitions for the mathematical concepts appearing in the definitions. In this paper, we consider a number of equivalent definitions for the notions of interval order and of proper interval order, and we examine how difficult it is to prove the equivalences of these definitions.

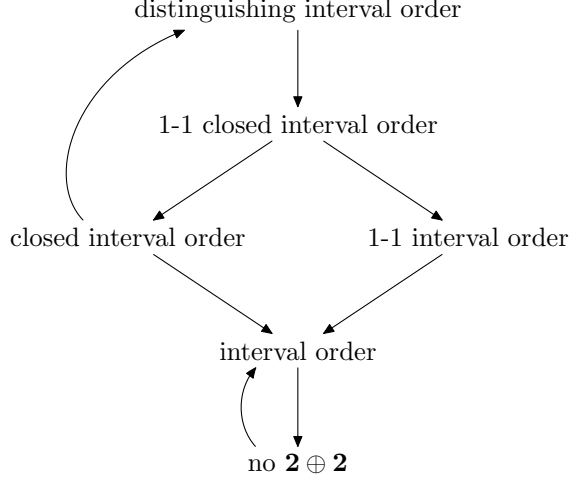


FIGURE 1. Implications about interval orders provable in RCA_0 .

There is no particular difficulty in coding a countable partial order in the weak base theory RCA_0 . The only point to note is that we consider only countable partial orders.

However the notion of interval order hinges on the notion of interval of a linear order, and the latter can be interpreted in different ways, leading to notions that are not necessarily equivalent in the weak base theory RCA_0 . We can define an interval of the linear order $\mathbf{L} = (L, \leq_L)$ to be a set $I \subseteq L$ which satisfies $\forall x, y \in I \forall z \in L (x \leq_L z \leq_L y \implies z \in I)$. Another possibility is to restrict our attention to closed intervals (this is often done in the literature about interval orders, e.g. in [22] this is done from the outset) and code them by pairs (a, b) of elements of L such that $a \leq_L b$ (obviously in this case $x \in L$ belongs to the interval if and only if $a \leq_L x \leq_L b$). If we apply the latter concept of interval we speak of a *closed interval representation* of the partial order. In defining interval orders there is a further subtlety, that turns out to be important in our study of the proof theoretic strength of various statements: i.e. we may require the map of the interval representation to be injective. Combining the two possible choices in each of the two cases we obtain four notions of interval order: interval order, 1-1 interval order, closed interval order, and 1-1 closed interval order. Another notion is obtained by further strengthening the definition of 1-1 closed interval order: a closed interval representation is a *distinguishing representation* if all endpoints of the closed intervals are distinct (see e.g. [22]). This leads to the notion of distinguishing interval order. In Section 2 we will give the precise definitions of these notions in RCA_0 .

The five notions introduced above are all equivalent, and we establish the axioms needed to show the equivalences among them and with the characterization provided by Characterization Theorem 1. (Notice that the proofs of the latter theorem in [9] and [22] can be easily carried out in ACA_0 : see Remark 3.8 below.)

We show that RCA_0 proves exactly the implications appearing in Figure 1 (where an arrow with origin in the node labeled A pointing towards the node labeled B represents the statement “every partial order which satisfies A satisfies B ”), or that can be obtained by composing arrows appearing in that diagram. In particular we obtain the following result about Characterization Theorem 1:

Theorem 1.1. *RCA_0 proves that a partial order is an interval order if and only if it does not contain $2 \oplus 2$.*

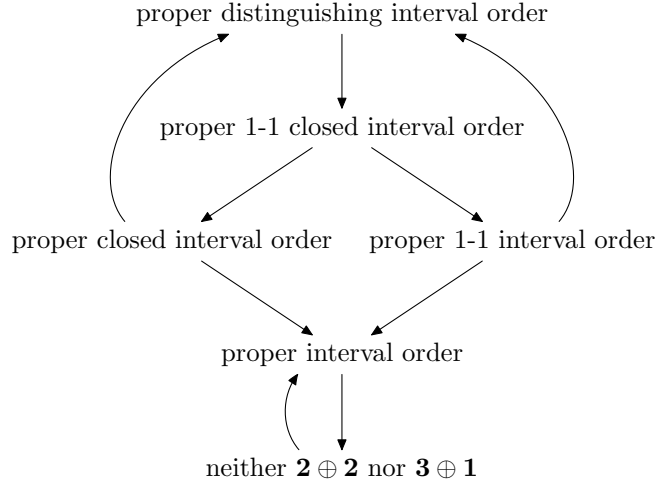


FIGURE 2. Implications about proper interval orders provable in RCA_0 .

The arrows pointing downwards (possibly diagonally) in Figure 1 either follow from the definitions or are straightforward to prove (these implications are collected in Theorem 2.13), while the two arrows pointing upwards will be proved in §4.

Figure 1 implies that in RCA_0 there are at most three distinct notions of interval order. In order of decreasing strength these are: closed interval order, 1-1 interval order, and interval order. In Section 5 we show that each of the missing implications is equivalent to WKL_0 . For the stronger notions of interval order we obtain the following reverse mathematics results about Characterization Theorem 1:

Theorem 1.2. *In RCA_0 the following are equivalent:*

- (i) WKL_0 ;
- (ii) a partial order is a 1-1 interval order if and only if it does not contain $\mathbf{2} \oplus \mathbf{2}$;
- (iii) a partial order is a closed interval order if and only if it does not contain $\mathbf{2} \oplus \mathbf{2}$;
- (iv) a partial order is a 1-1 closed interval order if and only if it does not contain $\mathbf{2} \oplus \mathbf{2}$;
- (v) a partial order is a distinguishing interval order if and only if it does not contain $\mathbf{2} \oplus \mathbf{2}$.

In particular this implies that RCA_0 does not prove that the equivalence between the three notions of interval order mentioned above.

Section 3 is devoted to a detailed analysis of the equivalences for finite partial orders; this analysis will be used in the proofs of the following sections.

When defining proper interval orders the same choices about intervals and injectivity are possible: we thus also have five different notions of proper interval order, plus the characterization provided by Characterization Theorem 2. We show that RCA_0 proves exactly the implications appearing in Figure 2, or that can be obtained by composing arrows appearing in that diagram. In particular we obtain the following result about Characterization Theorem 2:

Theorem 1.3. *RCA_0 proves that a partial order is a proper interval order if and only if it contains neither $\mathbf{2} \oplus \mathbf{2}$ nor $\mathbf{3} \oplus \mathbf{1}$.*

Figures 1 and 2 are similar, except that the latter includes one arrow whose analogous is missing from the former. Indeed within RCA_0 , a 1-1 interval order is necessarily a distinguishing interval order if we have a proper representation, but not in general.

Figure 2 implies that in RCA_0 there are at most two distinct notions of proper interval order, i.e. proper closed interval order and proper interval order. We show that the missing implication is equivalent to WKL_0 , even if we restrict ourselves to closed interval orders. For the stronger notions of interval order we obtain the following reverse mathematics results about Characterization Theorem 2:

Theorem 1.4. *In RCA_0 the following are equivalent:*

- (i) WKL_0 ;
- (ii) *a partial order is a proper 1-1 interval order if and only if it contains neither $\mathbf{2} \oplus \mathbf{2}$ nor $\mathbf{3} \oplus \mathbf{1}$;*
- (iii) *a partial order is a proper closed interval order if and only if it contains neither $\mathbf{2} \oplus \mathbf{2}$ nor $\mathbf{3} \oplus \mathbf{1}$;*
- (iv) *a partial order is a proper 1-1 closed interval order if and only if it contains neither $\mathbf{2} \oplus \mathbf{2}$ nor $\mathbf{3} \oplus \mathbf{1}$;*
- (v) *a partial order is a proper distinguishing interval order if and only if it contains neither $\mathbf{2} \oplus \mathbf{2}$ nor $\mathbf{3} \oplus \mathbf{1}$.*

In Section 6 the definitions and the arguments of Sections 2 through 5 are adapted to the case of proper interval orders, and all results about proper interval orders are proved. Some of the proofs are straightforward translations of the corresponding proofs for interval orders, while others exploit the properties of proper interval orders.

Our results are stated in terms of subsystems of second order arithmetic, but have corollaries that can be viewed as examples of computable mathematics in the style of [7]. Samples of these corollaries are the following, where we use standard terminology from computability theory:

Corollary 1.5. *For every computable partial order \mathbf{P} not containing $\mathbf{2} \oplus \mathbf{2}$ there exist a computable linear order \mathbf{L} and a computable function from P to intervals of \mathbf{L} witnessing that \mathbf{P} is an interval order.*

Corollary 1.6. *There exists a computable partial order \mathbf{P} not containing $\mathbf{2} \oplus \mathbf{2}$ such that for every computable linear order \mathbf{L} there is no computable function from P to closed intervals of \mathbf{L} witnessing that \mathbf{P} is a closed interval order.*

Corollary 1.7. *For every computable partial order \mathbf{P} not containing $\mathbf{2} \oplus \mathbf{2}$ there exist a low (resp. almost recursive) linear order \mathbf{L} and a low (resp. almost recursive) function from P to closed intervals of \mathbf{L} witnessing that \mathbf{P} is a distinguishing interval order.*

(The last Corollary follows from our results by the properties of ω -models of WKL_0 which appear in [20, §VIII.2].)

We assume some familiarity of the reader with subsystems of second order arithmetic, but the paper is self-contained as far as interval order theory is concerned.

From now on, when a definition or the statement of a result starts with the name of a subsystem of second order arithmetic in parenthesis, it means that the definition is given, or the statement provable, in that subsystem.

2. DEFINITIONS AND ELEMENTARY FACTS

Definition 2.1. (RCA_0) A *partial order* \mathbf{P} is a pair (P, \leq_P) where P is a set and $\leq_P \subseteq P \times P$ is reflexive, transitive and anti-symmetric. The partial order \mathbf{P} is a *linear order* if we have also $\forall p, q \in P (p \leq_P q \vee q \leq_P p)$.

Remark 2.2. If \mathbf{P} is a partial order then $P \subseteq \mathbb{N}$ and hence on P we have also the restriction of the usual order on the natural numbers. When there is danger of confusion we denote the latter by $\leq_{\mathbb{N}}$.

Definition 2.3. (RCA_0) If \mathbf{P} is a partial order we define the relations $<_P$ and \perp_P as follows:

$$\begin{aligned} p <_P q &\iff p \leq_P q \wedge p \neq q, \\ p \perp_P q &\iff p \not\leq_P q \wedge q \not\leq_P p. \end{aligned}$$

Sometimes it is convenient to use quasi-orders, which are defined by dropping the requirement of anti-symmetry from the definition of partial order. In particular we will be interested in linear quasi-orders.

Definition 2.4. (RCA_0) $\mathbf{P} = (P, \leq_P)$ is a *quasi-order* if $\leq_P \subseteq P \times P$ is reflexive and transitive. If we have also $\forall p, q \in P (p \leq_P q \vee q \leq_P p)$ we say that \mathbf{P} is a *linear quasi-order*.

Definition 2.5. (RCA_0) If \mathbf{P} is a quasi-order we define $<_P$ by

$$p <_P q \iff p \leq_P q \wedge p \not\leq_P q,$$

while no changes are needed in the definition of \perp_P . Furthermore we define \equiv_P by

$$p \equiv_P q \iff p \leq_P q \wedge p \leq_P q.$$

It is immediate to check in RCA_0 that if \mathbf{P} is a quasi-order then \equiv_P is an equivalence relation.

In our setting using (linear) quasi-orders in place of partial (resp. linear) orders is just a matter of convenience, as the following easy lemma shows.

Lemma 2.6. (RCA_0) Let \mathbf{P} be a quasi-order. Then there exist $P' \subseteq P$ and $f : P \rightarrow P'$ such that $\mathbf{P}' = (P', \leq_P)$ is a partial order and f is a surjective order-preserving function satisfying $f(p) = p$ for every $p \in P'$.

Furthermore, if \mathbf{P} is a linear quasi-order then \mathbf{P}' is a linear order.

Proof. Since $P \subseteq \mathbb{N}$ we can let

$$\begin{aligned} P' &= \{ p \in P \mid \forall q <_{\mathbb{N}} p \ q \not\equiv_P p \}; \\ f(p) &= \text{the } <_{\mathbb{N}}\text{-least } q \text{ such that } q \equiv_P p. \end{aligned}$$

□

We can now introduce the different notions of interval order.

Definition 2.7. (RCA_0) A partial order \mathbf{P} is an *interval order* if there exist a linear order \mathbf{L} and a set $F \subseteq P \times L$ such that, abbreviating $\{ x \in L \mid (p, x) \in F \}$ by $F(p)$ for every $p \in P$, we have:

- (i1) $F(p) \neq \emptyset$ and $\forall x, y \in F(p) \forall z \in L (x <_L z <_L y \implies z \in F(p))$ for all $p \in P$;
- (i2) $p <_P q \iff \forall x \in F(p) \forall y \in F(q) x <_L y$ for all $p, q \in P$.

\mathbf{P} is a *1-1 interval order* if we have also

- (i3) $F(p) \neq F(q)$ whenever $p \neq q$.

\mathbf{P} is a *closed interval order* if there exist a linear order \mathbf{L} and two functions $f_0, f_1 : P \rightarrow L$ such that:

- (c1) $f_0(p) \leq_L f_1(p)$ for all $p \in P$;
- (c2) $p <_P q \iff f_1(p) <_L f_0(q)$ for all $p, q \in P$.

\mathbf{P} is a *1-1 closed interval order* if we have also

- (c3) $f_0(p) \neq f_0(q)$ or $f_1(p) \neq f_1(q)$ whenever $p \neq q$.

\mathbf{P} is a *distinguishing interval order* if beside (c1–2) we have also

(c4) $f_i(p) \neq f_j(q)$ whenever $p \neq q$ or $i \neq j$.

It is immediate that if we set $F(p) = \{x \in L \mid f_0(p) \leq_L x \leq_L f_1(p)\}$, conditions (c1–3) are the translations of conditions (i1–3).

Remark 2.8. Lemma 2.6 implies that in the preceding definitions we can use linear quasi-orders in place of linear orders. Whenever it is convenient for the clarity of the exposition, we will use this fact without mentioning it explicitly.

Definition 2.9. (RCA_0) A partial order \mathbf{P} *does not contain* $\mathbf{2} \oplus \mathbf{2}$ if

$$\forall p_0, q_0, p_1, q_1 \in P (p_0 <_P q_0 \wedge p_1 <_P q_1 \implies p_0 \leq_P q_1 \vee p_1 \leq_P q_0).$$

Definition 2.10. (RCA_0) If \mathbf{P} is a partial order and $p \in P$ the *strict downward* and *upward closures* of p in P are the sets

$$p \uparrow^{\mathbf{P}} = \{q \in P \mid p <_P q\} \quad \text{and} \quad p \downarrow^{\mathbf{P}} = \{q \in P \mid q <_P p\}.$$

When \mathbf{P} is clear from the context we write $p \uparrow$ and $p \downarrow$.

The next lemma is a basic observation about partial orders not containing $\mathbf{2} \oplus \mathbf{2}$.

Lemma 2.11. (RCA_0) *If \mathbf{P} does not contain $\mathbf{2} \oplus \mathbf{2}$ then for every $p, q \in P$ we have either $p \uparrow \subseteq q \uparrow$ or $q \uparrow \subseteq p \uparrow$, and similarly either $p \downarrow \subseteq q \downarrow$ or $q \downarrow \subseteq p \downarrow$.*

Proof. If $p \uparrow \not\subseteq q \uparrow$ and $q \uparrow \not\subseteq p \uparrow$ let $p_1 \in p \uparrow \setminus q \uparrow$ and $q_1 \in q \uparrow \setminus p \uparrow$. Then p, p_1, q, q_1 show that \mathbf{P} contains $\mathbf{2} \oplus \mathbf{2}$.

The argument for the strict downward closures is similar. \square

The following lemma is useful to show that an interval order is actually a 1-1 interval order.

Lemma 2.12. *Suppose \mathbf{P} is an interval order such that*

$$\forall p, q \in P (p \neq q \implies p \uparrow \neq q \uparrow \vee p \downarrow \neq q \downarrow).$$

Then \mathbf{P} is a 1-1 interval order.

Proof. Let \mathbf{L} and F satisfy conditions (i1–2). We claim that F satisfies also (i3). Fix $p, q \in P$ with $p \neq q$. We have either $p \uparrow \neq q \uparrow$ or $p \downarrow \neq q \downarrow$. Without loss of generality, we may assume the former inequality holds and there exists $r \in p \uparrow \setminus q \uparrow$. Then $q \not\prec_P r$ and for some $x \in F(r)$ and $y \in F(q)$ we have $x \leq_L y$. On the other hand $p <_P r$ so that $z <_L x$ for all $z \in F(p)$. Hence $y \notin F(p)$ and $F(p) \neq F(q)$. \square

We now prove the “easy” arrows appearing in Figure 1.

Theorem 2.13. (RCA_0)

- (i) *Every distinguishing interval order is a 1-1 closed interval order.*
- (ii) *Every 1-1 (closed) interval order is a (closed) interval order.*
- (iii) *Every (1-1) closed interval order is a (1-1) interval order.*
- (iv) *Every interval order does not contain $\mathbf{2} \oplus \mathbf{2}$.*

Proof. The statements in (i) and (ii) follow immediately from the definitions (since condition (c4) implies condition (c3)).

For the statements in (iii), given \mathbf{L} , f_0 and f_1 as in the definition of closed interval order let

$$F = \{(p, x) \in P \times L \mid f_0(p) \leq_L x \leq_L f_1(p)\}.$$

To prove (iv), let L and F witness that \mathbf{P} is an interval order. Suppose towards a contradiction that $p_0, q_0, p_1, q_1 \in P$ are such that $p_0 <_P q_0$, $p_1 <_P q_1$, $p_0 \not\prec_P q_1$ and $p_1 \not\prec_P q_0$. The third condition implies the existence of $x, y \in L$ such that $x \in F(p_0)$, $y \in F(q_1)$, and $y \leq_L x$. Similarly by the fourth condition there exist x', y' such that $x' \in F(p_1)$, $y' \in F(q_0)$, and $y' \leq_L x'$. The first two conditions

imply respectively $x <_L y'$ and $x' <_L y$: using transitivity we have $x <_L x$, which is impossible. \square

3. FINITE INTERVAL ORDERS

We start by introducing one of the basic tools in the analysis of partial orders not containing $\mathbf{2} \oplus \mathbf{2}$. Within RCA_0 we can define it only for finite partial orders.

Definition 3.1. (RCA_0) Given a finite partial order \mathbf{P} , let $P^+ = \{p^+ \mid p \in P\}$, $P^- = \{p^- \mid p \in P\}$, and $P^* = P^+ \cup P^-$. Define a binary relation $\leq_{\mathbf{P}}^*$ on P^* as follows:

$$\begin{aligned} p^+ \leq_{\mathbf{P}}^* q^+ &\iff p \uparrow^{\mathbf{P}} \supseteq q \uparrow^{\mathbf{P}}; \\ p^- \leq_{\mathbf{P}}^* q^- &\iff p \downarrow^{\mathbf{P}} \supseteq q \downarrow^{\mathbf{P}}; \\ p^+ \leq_{\mathbf{P}}^* q^- &\iff p <_P q; \\ p^- \leq_{\mathbf{P}}^* q^+ &\iff q \not<_P p. \end{aligned}$$

$\mathbf{P}^* = (P^*, \leq_{\mathbf{P}}^*)$ is the *conjoint linear quasi-order associated to \mathbf{P}* . When \mathbf{P} is clear from the context we write \leq^* in place of $\leq_{\mathbf{P}}^*$.

The following lemma justifies the use of the words “linear quasi-order” in Definition 3.1.

Lemma 3.2. (RCA_0) *If \mathbf{P} is a finite partial order which does not contain $\mathbf{2} \oplus \mathbf{2}$ then \leq^* is a linear quasi-order.*

Moreover \mathbf{P}^ and the functions $p \mapsto p^-$, $p \mapsto p^+$ show that \mathbf{P} is a closed interval order.*

Proof. Reflexivity of \leq^* follows immediately from the definition. Using Lemma 2.11 it is also immediate that for every $x, y \in P^*$ we have $x \leq^* y$ or $y \leq^* x$.

It remains to show that \leq^* is transitive and to this end we need to consider eight cases. We tackle three of them, the others being trivial or similar to one of these:

- if $p^+ \leq^* q^+ \leq^* r^-$ then $p \uparrow \supseteq q \uparrow$ and $q <_P r$, i.e. $r \in q \uparrow$; therefore $r \in p \uparrow$ which means $p <_P r$ and hence $p^+ \leq^* r^-$;
- if $p^+ \leq^* q^- \leq^* r^+$ then $p <_P q$ and $r \not<_P q$; hence $q \in p \uparrow \setminus r \uparrow$ and, by Lemma 2.11, $p \uparrow \supset r \uparrow$ holds, so that $p^+ \leq^* r^+$;
- if $p^+ \leq^* q^- \leq^* r^-$ then $p <_P q$ and $q \downarrow \subseteq r \downarrow$, which imply $p <_P r$ and hence $p^+ \leq^* r^-$.

Since for every p we have $p^- \leq^* p^+$ (in fact $p^- <^* p^+$) condition (c1) of Definition 2.7 is satisfied. Condition (c2) follows immediately from the definition. \square

Remark 3.3. Notice that for all $p, q \in P$ we have $p^+ \not\equiv^* q^-$. In other words, each \equiv^* -equivalence class is contained in either P^+ or P^- .

Lemma 3.2 does not prove that \mathbf{P} is a distinguishing interval order, or even a 1-1 closed interval order: if $p, q \in P$ are distinct and such that $p \downarrow = q \downarrow$ and $p \uparrow = q \uparrow$ we have $p^- \equiv^* q^-$ and $p^+ \equiv^* q^+$. To obtain the stronger conclusions we can proceed as follows.

Definition 3.4. (RCA_0) Given a finite partial order \mathbf{P} which does not contain $\mathbf{2} \oplus \mathbf{2}$, let \mathbf{P}^* be the conjoint linear quasi-order associated to \mathbf{P} . A linear order (P^*, \leq_L) is *compatible with \mathbf{P}^** if

$$\forall x, y \in P^* (x <^* y \implies x <_L y).$$

Remark 3.5. Each (P^*, \leq_L) compatible with \mathbf{P}^* is defined by giving a linear order on each \equiv^* -equivalence class, and keeping the order between \equiv^* -inequivalent elements unchanged.

Lemma 3.6. (RCA_0) *If \mathbf{P} is a finite partial order which does not contain $\mathbf{2} \oplus \mathbf{2}$ then there exists a linear order compatible with \mathbf{P}^* .*

Proof. For example let

$$x \leq_L y \iff x <^* y \vee (x \equiv^* y \wedge x \leq_{\mathbb{N}} y).$$

\leq_L is a linear order compatible with \mathbf{P}^* . \square

Lemma 3.7. (RCA_0) *Any finite partial order which does not contain $\mathbf{2} \oplus \mathbf{2}$ is a distinguishing interval order.*

Proof. Let \mathbf{P} be a finite partial order which does not contain $\mathbf{2} \oplus \mathbf{2}$, and, by Lemma 3.6, \leq_L a linear order compatible with \mathbf{P}^* . By Lemma 3.2 and Remark 3.3 (P^*, \leq_L) and the functions $p \mapsto p^-$, $p \mapsto p^+$ witness that \mathbf{P} is a distinguishing interval order. \square

Combining Lemma 3.7 with Theorem 2.13 we obtain that RCA_0 proves the equivalence of the six characterizations of interval orders restricted in the case of finite partial orders.

Remark 3.8. The reader should notice that we carried out the discussion in this section only for finite partial orders, but the constructions and arguments apply also for infinite ones. However in the infinite case RCA_0 does not suffice to define \leq^* and we need to use ACA_0 . Indeed, arithmetical comprehension guarantees the existence of, say, the set of all pairs (p, q) such that $p \uparrow \supseteq q \uparrow$. Therefore we showed that ACA_0 proves the equivalence of the six characterizations of interval orders for countable partial orders.

Our goal is to obtain sharper results, in particular showing that all equivalences can be proved in WKL_0 (which is strictly weaker than ACA_0). We will in fact use the results of this section about finite partial orders to prove results about infinite partial orders without resorting to the full power of ACA_0 .

The following fact about the conjoint linear quasi-order will be useful in the proof of Theorem 4.2.

Lemma 3.9. *Let \mathbf{P}^* be the conjoint linear quasi-order associated to the finite partial order \mathbf{P} and let $p \in P$. Then:*

- *either p^- is a minimum in \mathbf{P}^* (i.e. $\forall x \in P^* p^- \leq^* x$) or there exists $q \in P$, $q \neq p$, such that q^+ is an immediate predecessor of p^- in \mathbf{P}^* (i.e. $x <^* p^-$ implies $x \leq^* q^+$ for all $x \in P^*$);*
- *either p^+ is a maximum in \mathbf{P}^* (i.e. $\forall x \in P^* x \leq^* p^+$) or there exists $q \in P$, $q \neq p$, such that q^- is an immediate successor of p^+ in \mathbf{P}^* (i.e. $p^+ <^* x$ implies $q^- \leq^* x$ for all $x \in P^*$).*

Proof. We prove the first statement (the second is proved similarly). Since \mathbf{P} and \mathbf{P}^* are finite, if p^- is not minimal in \mathbf{P}^* there exists $x \in P^*$ which is an immediate predecessor of p^- .

To show that $x = q^+$ for some q , it suffices to show that for every $r \in P$ with $r^- <^* p^-$ there exists $q \in P$ with $r^- \leq^* q^+ <^* p^-$. Indeed, $r^- <^* p^-$ means $r \downarrow \subsetneq p \downarrow$ and there exists $q \in p \downarrow \setminus r \downarrow$. Then $q \not\leq_P r$ and $q <_P p$ which imply $r^- \leq^* q^+$ and $q^+ <^* p^-$.

It is obvious that $q \neq p$, since $p^- <^* p^+$. \square

4. PROOFS IN RCA_0

We start this section with the quite simple proof of the upper upwards pointing arrow of Figure 1 is provable in RCA_0 .

Theorem 4.1. (RCA₀) *Every closed interval order is a distinguishing interval order.*

Proof. Let \mathbf{P} be a closed interval order and let \mathbf{L} , f_0 and f_1 witness this. Let $P^* = \{p^+, p^- \mid p \in P\}$ and $L' = L \cup P^*$ (we are assuming $L \cap P^* = \emptyset$).

We would like to define a linear order $\leq_{L'}$ on L' so that the maps $p \mapsto p^-$ and $p \mapsto p^+$ witness that \mathbf{P} is a distinguishing interval order. We first describe $\leq_{L'}$ informally: the restriction of $\leq_{L'}$ to L coincides with \leq_L , and p^+ and p^- are placed respectively “just above $f_1(p)$ ” and “just below $f_0(p)$ ”; if distinct p and q are such that $f_1(p) = f_1(q)$ then p^+ and q^+ are placed according to $\leq_{\mathbb{N}}$, and similarly for p^- and q^- when $f_0(p) = f_0(q)$; if $f_0(p) = f_1(q)$ then p^- is below q^+ .

To simplify the explicit definition of $\leq_{L'}$, we can exclude the elements not belonging to the range of the functions we have in mind, and therefore consider only the restriction of $\leq_{L'}$ to P^* . Thus we set, for every $p, q \in P$:

$$\begin{aligned} p^+ \leq_{L'} q^+ &\iff f_1(p) <_L f_1(q) \vee (f_1(p) = f_1(q) \wedge p \leq_{\mathbb{N}} q); \\ p^- \leq_{L'} q^- &\iff f_0(p) <_L f_0(q) \vee (f_0(p) = f_0(q) \wedge p \leq_{\mathbb{N}} q); \\ p^+ \leq_{L'} q^- &\iff f_1(p) <_L f_0(q); \\ p^- \leq_{L'} q^+ &\iff f_0(p) \leq_L f_1(q). \end{aligned}$$

It is left to the reader checking that $\mathbf{L}' = (P^*, \leq_{L'})$ is a linear order. We define $f'_0, f'_1 : P \rightarrow P^*$ by $f'_0(p) = p^-$ and $f'_1(p) = p^+$, and again we leave to the reader checking that conditions (c1-2) and (c4) of definition 2.7 hold. Therefore \mathbf{P} is a distinguishing interval order. \square

We now show that also the bottom upwards pointing arrow of Figure 1 is provable in RCA₀.

Theorem 4.2. (RCA₀) *Every partial order not containing $\mathbf{2} \oplus \mathbf{2}$ is an interval order.*

Proof. Let \mathbf{P} be a partial order not containing $\mathbf{2} \oplus \mathbf{2}$. Let $\{p_n \mid n > 0\}$ be an enumeration of P (notice that for notational convenience we start our enumeration from 1). If $s \in \mathbb{N}$ let $\mathbf{P}_s = (\{p_n \mid 0 < n \leq s\}, \leq_P)$ and let \mathbf{P}_s^* be the conjoint linear quasi-order associated to the finite partial order \mathbf{P}_s . We have $P_{s-1}^* \subset P_s^*$ and we can investigate which relations are preserved from \mathbf{P}_{s-1}^* to \mathbf{P}_s^* .

Claim 1. $x <_{s-1}^* y$ implies $x <_s^* y$ for every $x, y \in P_{s-1}^*$.

Proof. If exactly one of x and y is in P_{s-1}^+ (and the other is in P_{s-1}^-) the claim follows immediately from the definition of conjoint linear quasi-order. If $x, y \in P_{s-1}^+$, say $x = p_n^+$ and $y = p_m^+$, then $x <_{s-1}^* y$ means that $p_n \uparrow^{\mathbf{P}_{s-1}} \not\supseteq p_m \uparrow^{\mathbf{P}_{s-1}}$. Since $p_i \uparrow^{\mathbf{P}_s} \cap P_{s-1}^* = p_i \uparrow^{\mathbf{P}_{s-1}}$, $p_n \uparrow^{\mathbf{P}_s} \subseteq p_m \uparrow^{\mathbf{P}_s}$ cannot hold and, by Lemma 2.11 (which uses the hypothesis that \mathbf{P} does not contain $\mathbf{2} \oplus \mathbf{2}$), $p_n \uparrow^{\mathbf{P}_s} \supsetneq p_m \uparrow^{\mathbf{P}_s}$, i.e. $x <_s^* y$. The argument for the case $x, y \in P_{s-1}^-$ is similar. \square

On the other hand it is obvious that $x \equiv_{s-1}^* y$ does not imply $x \equiv_s^* y$, e.g. if $x = p_n^+$, $y = p_m^+$, $p_n \uparrow^{\mathbf{P}_{s-1}} = p_m \uparrow^{\mathbf{P}_{s-1}}$, $p_n <_{\mathbf{P}} p_s$, and $p_m \not<_{\mathbf{P}} p_s$. We say that x is *separated below at s* if for some y we have $x \equiv_{s-1}^* y$ and $x <_s^* y$. Analogously, x is *separated above at s* if for some y we have $x \equiv_{s-1}^* y$ and $y <_s^* x$.

Claim 2. *At most one \equiv_{s-1}^* -equivalence class contained in P_{s-1}^+ (recall Remark 3.3) contains elements separated at s (and the same for \equiv_{s-1}^* -equivalence classes contained in P_{s-1}^-).*

Proof. Notice that by Lemma 3.9 p_n^+ can be separated at s only if $x <_s^* p_s^- <_s^* y$ for some $x, y \equiv_{s-1}^* p_n^+$. By the previous claim, this can happen for the elements of at most one \equiv_{s-1}^* -equivalence class. \square

We define a linear quasi-order $\mathbf{L} = (L, \leq_L)$ where

$$L = \{ x_n^k \mid n \in \mathbb{N} \wedge n > 0 \wedge k \in \mathbb{Z} \wedge n \leq |k| \}.$$

If $s \in \mathbb{N}$ let $L_s = \{ x_n^k \in L \mid n \leq |k| \leq s \}$. We define \leq_L by stages, so that at stage $s \leq_L$ is defined on the finite set L_s and satisfies the following conditions:

- (i) the set $\{ x_n^s, x_n^{-s} \mid n \leq s \} \subseteq L_s$ is ordered by \leq_L according to \mathbf{P}_s^* , where x_n^s and x_n^{-s} replace respectively p_n^+ and p_n^- ;
- (ii) if $n < s$ then $x_n^{-s} <_L x_n^{-s+1}$ and $x_n^s >_L x_n^{s-1}$;
- (iii) if $n < s$ and $y \in L_{s-1}$ then neither $x_n^{-s} \leq_L y <_L x_n^{-s+1}$ nor $x_n^{s-1} <_L y \leq_L x_n^s$ hold.

An easy induction using (i) and (ii) yields $x_n^k <_L x_n^h$ if and only if $k <_{\mathbb{Z}} h$. Notice also that (i) and (iii) imply $x_n^k \neq_L x_m^h$ whenever $k \neq h$.

Since $L_0 = \emptyset$ at stage 0 there is nothing to do.

Let $s > 0$ and suppose we have defined \leq_L on L_{s-1} satisfying (i–iii). To define \leq_L on L_s it suffices to describe the position of the x_n^s 's and x_n^{-s} 's for $n \leq s$.

First consider x_n^s for $n < s$. If p_n^+ is not separated above at s then x_n^s is an immediate successor (among the elements of L_s) of x_n^{s-1} . If p_n^+ is separated above at s , fix p_m^+ which is separated below at s . By Claim 2 we have $p_m^+ \equiv_{s-1}^* p_n^+$, and hence by (i) $x_m^{s-1} \equiv_L x_n^{s-1}$. Let x_n^s be an immediate successor of x_n^{s-1} , which is an immediate successor of x_m^{s-1} (which, by the first clause of the present definition, is an immediate successor of $x_m^{s-1} \equiv_L x_n^{s-1}$). The position of x_n^{-s} for $n < s$ is established similarly: if p_n^- is not separated below at s then x_n^{-s} is an immediate predecessor of x_n^{-s+1} , otherwise fix p_m^- which is separated above at s and let x_n^{-s} be an immediate predecessor of x_n^{-s+1} , which is an immediate predecessor of x_m^{-s+1} .

If $p_s^+ \equiv_{P_s^*} p_n^+$ for some $n < s$ then set $x_s^s \equiv_L x_n^s$, and similarly if $p_s^- \equiv_{P_s^*} p_n^-$ for some $n < s$ set $x_s^{-s} \equiv_L x_n^{-s}$. If the previous case does not hold and p_s^+ is the maximum in \mathbf{P}_s^* then x_s^s is the maximum in L_s . Similarly if p_s^- is the minimum in \mathbf{P}_s^* then x_s^{-s} is the minimum in L_s . If the position of x_s^s is not yet determined, by Lemma 3.9 p_s^+ is the immediate predecessor in \mathbf{P}_s^* of some p_n^- with $n < s$: let x_s^s be the immediate predecessor of x_n^{-s} in L_s . Similarly if p_s^- is the immediate successor in \mathbf{P}_s^* of some p_n^+ with $n < s$, let x_s^{-s} be the immediate successor of x_n^s in L_s .

Notice that the latter part of the definition is compatible with the positions of x_s^s and x_s^{-s} given earlier in some cases (i.e. if some p_n^- or p_n^+ is separated at s) above: in fact if p_m^+ and p_n^+ are separated below and above, respectively, at s then p_s^- is an immediate successor in \mathbf{P}_s^* of p_m^+ (and similarly for the other case).

It is straightforward to check that \leq_L restricted to L_s satisfies (i–iii).

The definition of \mathbf{L} is thus complete. We need to define $F \subseteq P \times L$, and we would like to set

$$F = \{ (p_n, x_m^k) \mid \exists s \ x_n^{-s} \leq_L x_m^k \leq_L x_n^s \}.$$

To show the existence of F in RCA_0 , we need to prove that the Σ_1^0 formula appearing in the above definition is provably Δ_1^0 .

Claim 3. *If $t = \max(|k|, n)$ then $\exists s \ x_n^{-s} \leq_L x_m^k \leq_L x_n^s$ is equivalent to $x_n^{-t} \leq_L x_m^k \leq_L x_n^t$.*

Proof. One direction of the equivalence is obvious, so assume that $x_n^{-s} \leq_L x_m^k \leq_L x_n^s$ for some $s \neq t$. If $s < t$ the conclusion follows immediately from $x_n^{-t} <_L x_n^{-s}$ and $x_n^s <_L x_n^t$. If $s > t$ then $x_m^k \in L_{s-1}$ (because $m \leq |k| \leq t < s$) and $n < s$:

hence by (iii) we have $x_n^{-s+1} \leq_L x_m^k \leq_L x_n^{s-1}$. Repeating this argument we obtain $x_n^{-t} \leq_L x_m^k \leq_L x_n^t$. \square

Claim 3 shows that F exists. It is immediate that (i1) is satisfied, so we need only to check (i2). If $p_n <_P p_m$ then by (i) we have $x_n^s <_L x_m^{-s}$ for every $s \geq \max(n, m)$ and this easily implies $\forall x \in F(p_n) \forall y \in F(p_m) x <_L y$. If $p_n \not<_P p_m$ then $x_m^{-s} <_L x_n^s$ where $s = \max(n, m)$: since $x_n^s \in F(p_n)$ and $x_m^{-s} \in F(p_m)$, $\forall x \in F(p_n) \forall y \in F(p_m) x <_L y$ fails. \square

5. EQUIVALENCES WITH WKL_0

We first show that WKL_0 suffices to prove that the six characterizations of interval orders we introduced are equivalent.

Lemma 5.1. (WKL_0) *Every partial order not containing $\mathbf{2} \oplus \mathbf{2}$ is a distinguishing interval order.*

Proof. Let \mathbf{P} be a partial order not containing $\mathbf{2} \oplus \mathbf{2}$. By Lemma 3.7 we can assume P is infinite and let $\{p_n \mid n \in \mathbb{N}\}$ be a one-to-one enumeration of P . If $s \in \mathbb{N}$ let $\mathbf{P}_s = (\{p_n \mid n \leq s\}, \leq_P)$ and \mathbf{P}_s^* be the conjoint linear quasi-order associated to the finite partial order \mathbf{P}_s . \mathbf{P}_s^* is a linear quasi-order by Lemma 3.2 because \mathbf{P} , and hence the finite partial order \mathbf{P}_s , does not contain $\mathbf{2} \oplus \mathbf{2}$.

Let T be the set defined by setting $\sigma \in T$ if and only if σ is a finite sequence of length $\text{lh}(\sigma)$ such that for all $s, t < \text{lh}(\sigma)$:

- (1) $\sigma(s)$ is (the code for) a linear order (denoted by $\leq_{\sigma(s)}$) compatible with \mathbf{P}_s^* (see Definition 3.4);
- (2) if $s < t < \text{lh}(\sigma)$ then $\sigma(t)$ extends $\sigma(s)$, i.e. $x \leq_{\sigma(s)} y \iff x \leq_{\sigma(t)} y$ for all $x, y \in P_s^*$.

T exists by Δ_1^0 -comprehension. It is immediate that T is a tree. Since $\sigma(s)$ can assume only finitely many values (corresponding to the (codes of the) finitely many linear orders on the finite set P_s^*), T is bounded in the sense of [20, Definition IV.1.3]. By Lemma 3.6 for every s there exists a linear order compatible with \mathbf{P}_s^* . By taking its restrictions to P_t^* for $t < s$ we construct a sequence in T of length s . Thus T is infinite.

By Bounded König's Lemma, which is provable in WKL_0 ([20, Lemma IV.1.4]), T has an infinite path. This path is a sequence $\{\alpha(s) \mid s \in \mathbb{N}\}$ of (codes for) finite linear orders, each one extending the previous ones and such that $\alpha(s)$ is compatible with \mathbf{P}_s^* . If $x, y \in P^*$ let $x \leq_L y$ if and only if $x \leq_{\alpha(s)} y$ for any (or, equivalently, each) s with $x, y \in P_s^*$. (Notice that here we are considering P^* just as a set, without the ordering $\leq_{\mathbf{P}}^*$ which is not definable in WKL_0 .) \leq_L exists by Δ_1^0 -comprehension.

It is straightforward to check that (P^*, \leq_L) is a linear order and that conditions (c1–2) and (c4) are satisfied by the functions $p \mapsto p^-$, $p \mapsto p^+$ (because they are satisfied by each $\leq_{\alpha(s)}$, by the proof of Lemma 3.7). Hence \mathbf{P} is a distinguishing interval order. \square

Corollary 5.2. (WKL_0) *The five notions of interval order of Definition 2.7 and the property of not containing $\mathbf{2} \oplus \mathbf{2}$ are all equivalent.*

Proof. This follows from Theorem 2.13 and Lemma 5.1. \square

We now show that the implications that cannot be obtained by composing arrows appearing in Figure 1 are equivalent to WKL_0 . In particular these implications are not provable in RCA_0 .

The following well-known characterization of WKL_0 ([20, Lemma IV.4.4]) is useful.

Lemma 5.3. (RCA_0) *The following are equivalent:*

- (i) WKL_0 ;
- (ii) *if $f, g : \mathbb{N} \rightarrow \mathbb{N}$ are one-to-one functions such that $\forall n, m \ f(n) \neq g(m)$ then there exists a set X such that $\forall n (f(n) \in X \wedge g(n) \notin X)$.*

Lemma 5.4. (RCA_0) *If every interval order is a 1-1 interval order then WKL_0 holds.*

Proof. We will show that under our hypothesis (ii) of Lemma 5.3 holds. Fix one-to-one functions $f, g : \mathbb{N} \rightarrow \mathbb{N}$ such that $\forall n, m \ f(n) \neq g(m)$. We want to find a set X such that $\forall n (f(n) \in X \wedge g(n) \notin X)$.

We define a partial order \leq_P on the set $P = \bigcup_{k \in \mathbb{N}} P_k$, where $P_k = \{a_k, b_k\} \cup \{c_k^n \mid n \in \mathbb{N}\}$ for each k . If $p \in P_k$ and $q \in P_h$ with $k \neq h$ we set $p \leq_P q$ if and only if $k <_{\mathbb{N}} h$. The elements of each P_k are pairwise \leq_P -incomparable with the following exceptions:

- if n is such that $f(n) = k$ then $c_k^n <_P a_k <_P c_k^{n+1}$;
- if n is such that $g(n) = k$ then $c_k^n <_P b_k <_P c_k^{n+1}$.

Notice that our hypothesis on f and g imply that for each k at most one of the two possibilities occurs, and for at most one n . \leq_P can be defined within RCA_0 .

Let $\mathbf{P} = (P, \leq_P)$: it is immediate that \mathbf{P} does not contain $\mathbf{2} \oplus \mathbf{2}$. By Theorem 4.2 \mathbf{P} is an interval order and by our hypothesis \mathbf{P} is a 1-1 interval order. Hence there exist a linear order $\mathbf{L} = (L, \leq_L)$ and $F \subseteq P \times L$ satisfying conditions (i1–3) of Definition 2.7. Let $\varphi(k)$ and $\psi(k)$ be the Π_1^0 formulas

$$F(a_k) \subseteq F(b_k) \quad \text{and} \quad F(b_k) \subseteq F(a_k),$$

respectively. Since (i3) holds (i.e. F is one-to-one) we have $\forall k \neg(\varphi(k) \wedge \psi(k))$ and we are in the hypothesis of Π_1^0 -separation ([20, Exercise IV.4.8]), which is provable in RCA_0 : hence there exists a set X satisfying

$$\forall k ((\varphi(k) \implies k \in X) \wedge (\psi(k) \implies k \notin X)).$$

We claim that X satisfies also $\forall n (f(n) \in X \wedge g(n) \notin X)$, thus completing the proof. To this end it suffices to show that $\exists n \ f(n) = k$ implies $\varphi(k)$ and $\exists n \ g(n) = k$ implies $\psi(k)$.

We prove only the first of these implications, the second being similar. Suppose n is such that $f(n) = k$: then $c_k^n <_P a_k <_P c_k^{n+1}$, $c_k^n \not\leq_P b_k$, and $b_k \not\leq_P c_k^{n+1}$. The last two conditions and (i2) imply the existence of $x \in F(c_k^n)$, $x' \in F(b_k)$, $y \in F(c_k^{n+1})$, and $y' \in F(b_k)$ such that $x' \leq_L x$ and $y \leq_L y'$. By the first condition and (i2), for all $z \in F(a_k)$ we have $x <_L z <_L y$, and hence $x' <_L z <_L y'$. Now we use (i1), obtaining $F(a_k) \subseteq F(b_k)$, i.e. $\varphi(k)$. \square

Lemma 5.5. (RCA_0) *If every 1-1 interval order is a closed interval order then WKL_0 holds.*

Proof. Again we will show that under our hypothesis (ii) of Lemma 5.3 holds and we fix one-to-one functions $f, g : \mathbb{N} \rightarrow \mathbb{N}$ such that $\forall n, m \ f(n) \neq g(m)$. We want to find X such that $\forall n (f(n) \in X \wedge g(n) \notin X)$.

We define a partial order \leq_P on the set $P = \bigcup_{k \in \mathbb{N}} P_k$, where $P_k = \{a_k, b_k, c_k\} \cup \{d_k^n \mid n \in \mathbb{N}\}$ for each k . As in the previous proof, if $p \in P_k$ and $q \in P_h$ with $k \neq h$ we set $p \leq_P q$ if and only if $k <_{\mathbb{N}} h$. Within each P_k we have:

- $a_k \perp_P b_k$, $a_k \perp_P c_k$, and $c_k <_P b_k$;
- if $f(n) \neq k \neq f(m)$ and $g(n) \neq k \neq g(m)$ then $d_k^n <_P d_k^m$ if and only if $n <_{\mathbb{N}} m$;
- if $f(n) \neq k$ and $g(n) \neq k$ then $a_k, b_k, c_k <_P d_k^n$;
- if $f(n) = k$ and $m \neq n$ then $a_k, c_k <_P d_k^n <_P d_k^m$ and $b_k \perp_P d_k^n$;

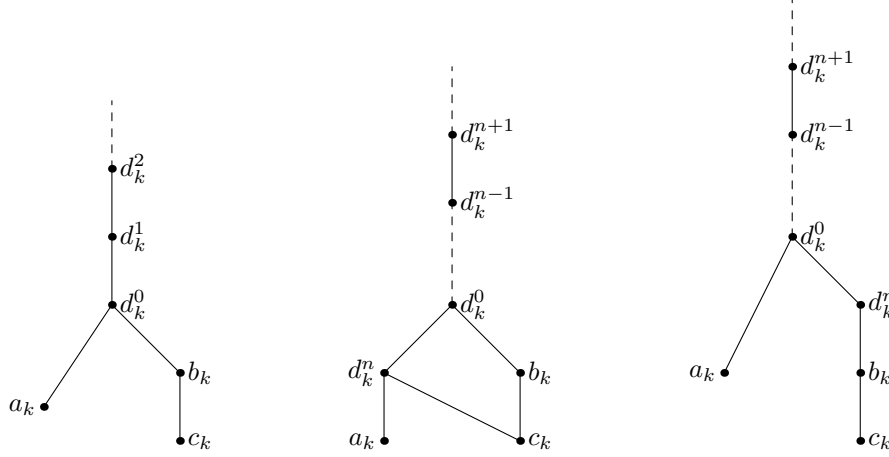


FIGURE 3. The three cases of \leq_P restricted to P_k in the proof of Lemma 5.5: from left to right $\forall n \ f(n) \neq k \neq g(n)$, $f(n) = k$, and $g(n) = k$.

- if $g(n) = k$ and $m \neq n$ then $b_k, c_k <_P d_k^m <_P d_k^n$ and $a_k \perp_P d_k^m$.

Figure 3 contains the Hasse diagram of the most significant part of the restriction of \leq_P to P_k in the three possible cases.

\leq_P can be defined in RCA_0 . Let $\mathbf{P} = (P, \leq_P)$.

Claim 1. \mathbf{P} is a 1-1 interval order.

Proof. It is easy to check that \mathbf{P} does not contain $\mathbf{2} \oplus \mathbf{2}$ and hence it is an interval order by Theorem 4.2. By Lemma 2.12 to prove the claim it suffices to show that

$$\forall p, q \in P \ (p \neq q \implies p \uparrow \neq q \uparrow \vee p \downarrow \neq q \downarrow).$$

Fix $p, q \in P$ with $p \neq q$. If $p <_P q$ or $q <_P p$ then both $p \uparrow \neq q \uparrow$ and $p \downarrow \neq q \downarrow$ hold. If $p \perp_P q$ then $p, q \in P_k$ for some k , and we consider the different possibilities. In each case we exhibit an element of P witnessing either $p \uparrow \neq q \uparrow$ or $p \downarrow \neq q \downarrow$: $c_k \in b_k \downarrow \setminus a_k \downarrow$, $b_k \in c_k \uparrow \setminus a_k \uparrow$, if $f(n) = k$ then $a_k \in d_k^n \downarrow \setminus b_k \downarrow$, and if $g(n) = k$ then $b_k \in d_k^n \downarrow \setminus a_k \downarrow$. \square

By our hypothesis \mathbf{P} is a closed interval order and there exist a linear order \mathbf{L} and $f_0, f_1 : P \rightarrow L$ satisfying (c1-2). Let $X = \{k \mid f_1(a_k) \leq_L f_1(b_k)\}$. To complete the proof we need to check that $f(n) \in X$ and $g(n) \notin X$ for every n . If $k = f(n)$ then $f_1(a_k) <_L f_0(d_k^n) \leq_L f_1(b_k)$ and $k \in X$. If $k = g(n)$ then $f_1(b_k) <_L f_0(d_k^n) \leq_L f_1(a_k)$ and $k \notin X$. \square

We summarize our results in the following theorem (a few more implications equivalent to WKL_0 can be stated using the information contained in Figure 1, Corollary 5.2, and Lemmas 5.4 and 5.5).

Theorem 5.6. (RCA_0) *The following are equivalent:*

- (i) WKL_0 ;
- (ii) every partial order not containing $\mathbf{2} \oplus \mathbf{2}$ is a 1-1 interval order;
- (iii) every interval order is a 1-1 interval order;
- (iv) every 1-1 interval order is a distinguishing interval order;
- (v) every 1-1 interval order is a closed interval order.

Proof. The forward direction, i.e. the fact that (i) implies each of (ii)–(v), is a consequence of Corollary 5.2.

The implication (ii) \implies (iii) follows from Theorem 2.13(iv). Lemma 5.4 shows that (iii) implies (i). The implication (iv) \implies (v) is immediate by Theorem 2.13. Lemma 5.5 shows that (v) implies (i). \square

6. PROPER INTERVAL ORDERS

In this section we deal with proper interval orders. Throughout most of the section we point out the changes needed in the definitions and proofs of §2–5. However, Theorem 6.16 is new, because its statement without “proper” is false by Lemma 5.5. The proof of Lemma 6.21 is also new, because the interval order used in the proof of Lemma 5.4 is not proper.

We start with the definitions and elementary facts corresponding to Section 2.

Definition 6.1. (RCA₀) A partial order \mathbf{P} is a *proper interval order* if there exist a linear order \mathbf{L} and a set $F \subseteq P \times L$ such that (i1–2) of Definition 2.7 hold and moreover:

(i4) $F(p) \subseteq F(q)$ implies $F(p) = F(q)$ for all $p, q \in P$.

\mathbf{P} is a *proper 1-1 interval order* if (i3) of Definition 2.7 holds as well.

\mathbf{P} is a *proper closed interval order* if there exist a linear order \mathbf{L} and functions $f_0, f_1 : P \rightarrow L$ such that (c1–2) of Definition 2.7 hold and moreover:

(c5) $f_0(p) <_L f_0(q)$ if and only if $f_1(p) <_L f_1(q)$ for all $p, q \in P$.

\mathbf{P} is a *proper 1-1 closed interval order* if (c3) of Definition 2.7 holds as well. \mathbf{P} is a *proper distinguishing interval order* if beside (c1–2) and (c5) we have also (c4).

Definition 6.2. (RCA₀) A partial order \mathbf{P} *does not contain* $\mathbf{3} \oplus \mathbf{1}$ if

$$\forall p_0, p_1, p_2, q \in P (p_0 <_P p_1 <_P p_2 \implies p_0 \leq_P q \vee q \leq_P p_2).$$

Lemma 6.3. (RCA₀) If \mathbf{P} does not contain $\mathbf{3} \oplus \mathbf{1}$ then for every $p, q \in P$ we have either $p \downarrow \subseteq q \downarrow$ or $p \uparrow \subseteq q \uparrow$.

Proof. Towards a contradiction assume that $p \downarrow \not\subseteq q \downarrow$ and $p \uparrow \not\subseteq q \uparrow$. If $p_0 \in p \downarrow \setminus q \downarrow$ and $p_2 \in p \uparrow \setminus q \uparrow$, then p_0, p, p_2, q witness that \mathbf{P} contains $\mathbf{3} \oplus \mathbf{1}$. \square

Theorem 6.4. (RCA₀)

- (i) Every proper (distinguishing) (1-1) (closed) interval order is a (distinguishing) (1-1) (closed) interval order.
- (ii) Every proper distinguishing interval order is a proper 1-1 closed interval order.
- (iii) Every proper 1-1 (closed) interval order is a proper (closed) interval order.
- (iv) Every proper (1-1) closed interval order is a proper (1-1) interval order.
- (v) Every proper interval order contains neither $\mathbf{2} \oplus \mathbf{2}$ nor $\mathbf{3} \oplus \mathbf{1}$.

Proof. Statement (i) is immediate from the definitions. The statements in (ii–iv) are proved exactly as the corresponding statements in Theorem 2.13.

To prove (v) let \mathbf{P} be a proper interval order: by (i) above \mathbf{P} is an interval order and by Theorem 2.13(iv) \mathbf{P} does not contain $\mathbf{2} \oplus \mathbf{2}$.

To show that \mathbf{P} does not contain $\mathbf{3} \oplus \mathbf{1}$ let L and F witness that \mathbf{P} is a proper interval order, and suppose towards a contradiction that $p_0, p_1, p_2, q \in P$ are such that $p_0 <_P p_1 <_P p_2$, $p_0 \not\leq_P q$ and $q \not\leq_P p_2$. The second condition implies the existence of $x, y \in L$ such that $x \in F(p_0)$, $y \in F(q)$, and $y \leq_L x$. Similarly by the third condition there exist y', x' such that $y' \in F(q)$, $x' \in F(p_2)$, and $x' \leq_L y'$. For every $z \in F(p_1)$ the first condition implies $x <_L z <_L x'$: this implies on one hand $y, y' \notin F(p)$, and on the other hand $y <_L z <_L y'$ and hence $z \in F(q)$ by (i1), for all $z \in F(p_1)$. Therefore $F(p_1) \subsetneq F(q)$, contradicting condition (i4). \square

We now analyze finite partial orders containing neither $\mathbf{2} \oplus \mathbf{2}$ nor $\mathbf{3} \oplus \mathbf{1}$, imitating what we did in Section 3.

Definition 6.5. (RCA₀) Given a finite partial order \mathbf{P} let $P^\# = P^*$ be defined as in Definition 3.1. Define a binary relation $\leq_{\mathbf{P}}^\#$ on $P^\#$ as follows:

$$\begin{aligned} p^+ \leq_{\mathbf{P}}^\# q^+ &\iff p \uparrow^{\mathbf{P}} \supseteq q \uparrow^{\mathbf{P}} \vee (p \uparrow^{\mathbf{P}} = q \uparrow^{\mathbf{P}} \wedge p \downarrow^{\mathbf{P}} \subseteq q \downarrow^{\mathbf{P}}); \\ p^- \leq_{\mathbf{P}}^\# q^- &\iff p \downarrow^{\mathbf{P}} \supseteq q \downarrow^{\mathbf{P}} \vee (p \downarrow^{\mathbf{P}} = q \downarrow^{\mathbf{P}} \wedge p \uparrow^{\mathbf{P}} \supseteq q \uparrow^{\mathbf{P}}); \\ p^+ \leq_{\mathbf{P}}^\# q^- &\iff p <_P q; \\ p^- \leq_{\mathbf{P}}^\# q^+ &\iff q <_P p. \end{aligned}$$

$\mathbf{P}^\# = (P^\#, \leq_{\mathbf{P}}^\#)$ is the *proper conjoint linear quasi-order associated to \mathbf{P}* . When \mathbf{P} is clear from the context we write $\leq^\#$ in place of $\leq_{\mathbf{P}}^\#$.

Remark 6.6. Notice that $\leq_{\mathbf{P}}^\#$ and $\leq_{\mathbf{P}}^*$ are defined on the same set. It is immediate that $\leq_{\mathbf{P}}^\# \subseteq \leq_{\mathbf{P}}^*$, and in general equality does not hold: in fact if $p \uparrow^{\mathbf{P}} = q \uparrow^{\mathbf{P}}$ it is always the case that $p^+ \leq_{\mathbf{P}}^* q^+$, while $p^+ \leq_{\mathbf{P}}^\# q^+$ fails when $p \downarrow^{\mathbf{P}} \not\subseteq q \downarrow^{\mathbf{P}}$.

The following lemma justifies the use of the words “linear quasi-order” in Definition 6.5.

Lemma 6.7. (RCA₀) *If \mathbf{P} is a finite partial order which does not contain $2 \oplus 2$ then $\leq^\#$ is a linear quasi-order.*

Moreover, if \mathbf{P} does not contain $3 \oplus 1$ then $\mathbf{P}^\#$ and the functions $p \mapsto p^-$, $p \mapsto p^+$ show that \mathbf{P} is a proper closed interval order.

Proof. The proofs that $\leq^\#$ is a linear quasi-order and that the functions $p \mapsto p^-$, $p \mapsto p^+$ witness that \mathbf{P} is a closed interval order are identical to the same proofs for \leq^* in Lemma 3.2. Hence we need only to show that condition (c5) of Definition 6.1 is met, i.e. that $p^- <^\# q^-$ if and only if $p^+ <^\# q^+$ for all $p, q \in P$.

Suppose $p, q \in P$ are such that $p^- <^\# q^-$ holds. Then either $p \downarrow \subsetneq q \downarrow$ or $p \downarrow = q \downarrow$ and $p \uparrow \supsetneq q \uparrow$. In the first case Lemma 6.3 implies that $q \uparrow \subseteq p \uparrow$; even if $q \uparrow = p \uparrow$ we have $q^+ \not\leq^\# p^+$ (because $q \downarrow \not\subseteq p \downarrow$) and hence $p^+ <^\# q^+$. In the second case $p^+ <^\# q^+$ is immediate.

The reverse implication is proved similarly. \square

Remark 6.8. Remark 3.3 applies also to $\leq^\#$, i.e. each $\equiv^\#$ -equivalence class is contained in either P^+ or P^- . Moreover $p^+ \equiv^\# q^+$ if and only if $p \uparrow^{\mathbf{P}} = q \uparrow^{\mathbf{P}}$ and $p \downarrow^{\mathbf{P}} = q \downarrow^{\mathbf{P}}$, if and only if $p^- \equiv^\# q^-$. Therefore the $\equiv^\#$ -equivalence classes contained in P^+ are paired in a straightforward way with those contained in P^- .

Definition 6.9. (RCA₀) Given a finite partial order \mathbf{P} which contains neither $2 \oplus 2$ nor $3 \oplus 1$, let $\mathbf{P}^\#$ be the proper conjoint linear quasi-order associated to \mathbf{P} . A linear order $(P^\#, \leq_L)$ is *compatible with $\mathbf{P}^\#$* if

$$\begin{aligned} \forall x, y \in P^\# (x <^\# y \implies x <_L y), \\ \forall p, q \in P (p \neq q \wedge p^+ \equiv^\# q^+ \wedge p^+ <_L q^+ \implies p^- <_L q^-), \quad \text{and} \\ \forall p, q \in P (p \neq q \wedge p^- \equiv^\# q^- \wedge p^- <_L q^- \implies p^+ <_L q^+). \end{aligned}$$

(Actually the second and third conditions imply each other.)

Remark 6.10. Defining $(P^\#, \leq_L)$ compatible with $\mathbf{P}^\#$ means defining a linear order on each $\equiv^\#$ -equivalence class, and keeping the order between $\equiv^\#$ -inequivalent elements unchanged. Moreover we require that the linear orders on the $\equiv^\#$ -equivalence classes containing p^+ and p^- are the same.

Lemma 6.11. (RCA₀) *If \mathbf{P} is a finite partial order which contains neither $2 \oplus 2$ nor $3 \oplus 1$ then there exists a linear order compatible with $\mathbf{P}^\#$.*

Proof. For example let

$$x \leq_L y \iff x <^\# y \vee (x \equiv^\# y \wedge x \leq_{\mathbb{N}} y).$$

\leq_L is a linear order compatible with $\mathbf{P}^\#$. \square

Lemma 6.12. (RCA_0) *Any finite partial order which contains neither $\mathbf{2} \oplus \mathbf{2}$ nor $\mathbf{3} \oplus \mathbf{1}$ is a proper distinguishing interval order.*

Proof. Let \mathbf{P} be a finite partial order which contains neither $\mathbf{2} \oplus \mathbf{2}$ nor $\mathbf{3} \oplus \mathbf{1}$, and, by Lemma 6.11, \leq_L a linear order compatible with $\mathbf{P}^\#$. Then $(P^\#, \leq_L)$ and the functions $p \mapsto p^-$, $p \mapsto p^+$ show that \mathbf{P} is a proper distinguishing interval order. Indeed if $p \neq q$ and, say, $p^+ \equiv^\# q^+$ then we have also $p^- \equiv^\# q^-$: if $p^+ <_L q^+$ then the second condition of Definition 6.9 implies $p^- <_L q^-$. \square

Combining Lemma 6.12 with Theorem 6.4 we obtain that RCA_0 proves the equivalence of the six characterizations of proper interval orders in the finite case.

Remark 6.13. Remark 3.8 applies also to what we have done with $\leq^\#$ in the previous Lemmas, and we can conclude that ACA_0 suffices to prove the equivalence of the six characterizations of proper interval orders for countable partial orders.

As with interval orders, we will obtain sharper results also for proper interval orders, in particular showing that all equivalences can be proved in WKL_0 .

Remark 6.14. Notice that Lemma 3.9 does not hold with $\mathbf{P}^\#$ in place of \mathbf{P}^* . If $P = \{p, q, r\}$ is ordered by \leq_P as $\mathbf{2} \oplus \mathbf{1}$ (i.e. the only nonreflexive relation is $p <_P q$) then $p^- <^\# r^- <^\# p^+ <^\# q^- <^\# r^+ <^\# q^+$.

Now we show that the upwards pointing implications of Figure 2 are provable in RCA_0 , much as we did with Figure 1 in Section 4.

Theorem 6.15. (RCA_0) *Every proper closed interval order is a proper distinguishing interval order.*

Proof. We can repeat the proof of Theorem 4.1. One needs only to check that the construction preserves properness. We leave this to the reader. \square

As already noticed, the next Theorem has no counterpart for arbitrary interval orders.

Theorem 6.16. (RCA_0) *Every proper 1-1 interval order is a proper closed interval order.*

Proof. Let $\mathbf{L} = (L, F)$ witness that the partial order \mathbf{P} is a proper 1-1 interval order.

Claim 1. *For all $p, q \in P$ the following are equivalent:*

- (1) $p = q \vee \exists x, y \in L (x \in F(p) \setminus F(q) \wedge y \in F(q) \wedge x <_L y)$;
- (2) $\forall x, y \in L (x \in F(p) \setminus F(q) \wedge y \in F(q) \implies x <_L y)$.

Proof. First assume that (1) holds and (2) fails. Since $p = q$ implies (2), there exist $x, y, x', y' \in L$ with $x, x' \in F(p) \setminus F(q)$, $y, y' \in F(q)$, $x <_L y$ and $y' <_L x'$. Let $z \in F(q)$: we have neither $z \leq_L x$ (because $x \notin F(q)$) nor $x' \leq_L z$ (because $x' \notin F(q)$). Hence $x <_L z <_L x'$ and $F(q) \subseteq F(p)$. Since it is immediate that $F(q) \neq F(p)$, we are contradicting condition (i4) in definition 6.1.

Now assume (2) holds and (1) fails, so that in particular $p \neq q$ and hence $F(p) \neq F(q)$ because condition (i3) holds. If $F(p) \setminus F(q) = \emptyset$ then $F(q) \subseteq F(p)$ and we are again contradicting (i4). Therefore we can choose $x \in F(p) \setminus F(q)$ and $y \in F(q)$: (2) implies $x <_L y$ and then we have (1), against our assumption. \square

Obviously (1) is Σ_1^0 and (2) is Π_1^0 . We denote either of them by $\varphi(p, q)$: φ is a provably Δ_1^0 formula and we can use it in the comprehension scheme. The following two claims about φ are useful.

Claim 2. $\varphi(p, q)$ implies $q \uparrow \subseteq p \uparrow$ and $p \downarrow \subseteq q \downarrow$.

Proof. Let $r \in q \uparrow$: to show $r \in p \uparrow$, i.e. $p <_P r$, by (i2) it suffices to show that $x <_L z$ for all $x \in F(p)$ and $z \in F(r)$. If $x \in F(q)$ this follows from $q <_P r$. If $x \in F(p) \setminus F(q)$ let $y \in F(q)$: we have $x <_L y <_L z$ and we are done.

The proof that $p \downarrow \subseteq q \downarrow$ is even simpler. \square

Claim 3. For every $p, q \in P$ either $\varphi(p, q)$ or $\varphi(q, p)$ holds.

Proof. When $p = q$ the claim is obvious, so we assume $p \neq q$. Then $F(p) \neq F(q)$ by (i3) and by (i4) $F(p) \setminus F(q)$ and $F(q) \setminus F(p)$ are both nonempty. Let $x \in F(p) \setminus F(q)$ and $y \in F(q) \setminus F(p)$: if $x <_L y$ then $\varphi(p, q)$ holds, if $y <_L x$ then we have $\varphi(q, p)$. \square

Let $P^\# = P^+ \cup P^-$ and define $\leq_{L'}$ by

$$\begin{aligned} p^+ \leq_{L'} q^+ &\iff \varphi(p, q); \\ p^- \leq_{L'} q^- &\iff \varphi(p, q); \\ p^+ \leq_{L'} q^- &\iff p <_P q; \\ p^- \leq_{L'} q^+ &\iff q \not<_P p. \end{aligned}$$

Reflexivity of $\leq_{L'}$ is immediate from the fact that $\varphi(p, p)$ holds for every p . To check transitivity start by noticing that using (2) it is immediate that $\varphi(p, q)$ and $\varphi(q, r)$ imply $\varphi(p, r)$. This gives two of the eight cases. The other four cases where some hypothesis is of the form $\varphi(p, q)$, are easily handled using Claim 2. Only two cases are left:

- if $p^+ \leq_{L'} q^- \leq_{L'} r^+$ then $p <_P q$ and $r \not<_P q$. Thus there exist $z \in F(r)$ and $y \in F(q)$ with $y \leq_L z$. Since $p \neq r$ we can pick $x \in F(p) \setminus F(r)$: we have $x <_L y$ and hence $x <_L z$. Therefore $\varphi(p, r)$ and $p^+ \leq_{L'} r^+$;
- if $p^- \leq_{L'} q^+ \leq_{L'} r^-$ then $q \not<_P p$ and $q <_P r$. Let $x \in F(p)$ and $y \in F(q)$ be such that $x \leq_L y$. Since $p \neq r$ we can choose $z \in F(r) \setminus F(p)$: $x <_L z$ follows immediately and hence we have that $\varphi(r, p)$ does not hold. By Claim 3 we have $\varphi(p, r)$ and $p^- \leq_{L'} r^-$.

The fact that $(P^\#, \leq_{L'})$ is linear follows immediately from the definition and Claim 3.

Define $f_0, f_1 : P \rightarrow P^\#$ as usual by $f_0(p) = p^-$ and $f_1(p) = p^+$. Conditions (c1-2) and (c5) follow immediately from the definition of $\leq_{L'}$. Therefore \mathbf{P} is a proper closed interval order. \square

Remark 6.17. The reader may have noticed the construction of the proof of Theorem 6.16 satisfies also condition (c4). Therefore the proof actually shows that RCA_0 suffices to prove that every proper 1-1 interval order is a proper distinguishing interval order. This result is also obtained combining the statements of Theorems 6.16 and 6.15.

Theorem 6.18. (RCA_0) Every partial order which contains neither $\mathbf{2} \oplus \mathbf{2}$ nor $\mathbf{3} \oplus \mathbf{1}$ is a proper interval order.

Proof. The proof follows the pattern of the proof of Theorem 4.2: throughout the proof we replace \mathbf{P}_s^* with $\mathbf{P}_s^\#$, the proper conjoint linear quasi-order associated to \mathbf{P}_s . We point out only the spots where differences occur.

To prove the analogous of Claim 1 we need to consider the case of $n, m < s$ such that $p_n^+ <_{s-1}^\# p_m^+$ because $p_n \uparrow^{\mathbf{P}_{s-1}} = p_m \uparrow^{\mathbf{P}_{s-1}}$ and $p_n \downarrow^{\mathbf{P}_{s-1}} \subsetneq p_m \downarrow^{\mathbf{P}_{s-1}}$. Beside

Lemma 2.11, also Lemma 6.3 (which uses the hypothesis that \mathbf{P} does not contain $\mathbf{3} \oplus \mathbf{1}$) is needed here: since $p_m \downarrow^{\mathbf{P}_s} \not\subseteq p_n \downarrow^{\mathbf{P}_s}$ we have $p_m \uparrow^{\mathbf{P}_s} \subseteq p_n \uparrow^{\mathbf{P}_s}$ and therefore $p_m \uparrow^{\mathbf{P}_s} \not\supseteq p_n \uparrow^{\mathbf{P}_s}$ cannot occur. Hence $p_n^+ <_s^\# p_m^+$.

The analogous of Claim 2 states that at most two $\equiv_{s-1}^\#$ -equivalence class contained in P_{s-1}^+ contain elements separated at s , and the same for $\equiv_{s-1}^\#$ -equivalence classes contained in P_{s-1}^- .

The definition of \leq_L on L_s requires considering a few more possible situations. When $n < s$ and p_n^+ is separated above at s , fix p_m^+ separated below at s with $p_m^+ \equiv_{s-1}^\# p_n^+$ and hence $x_m^{s-1} \equiv_L x_n^{s-1}$. If $p_n \uparrow^{\mathbf{P}_s} \not\subseteq p_m \uparrow^{\mathbf{P}_s}$ then no changes are needed, but now it might happen that $p_n \uparrow^{\mathbf{P}_s} = p_m \uparrow^{\mathbf{P}_s}$ (because $p_n \downarrow^{\mathbf{P}_s} \not\supseteq p_m \downarrow^{\mathbf{P}_s}$ forces $p_m^+ <_s^\# p_n^+$). In the latter case x_n^s is an immediate successor of x_m^s , which by the other clauses in the definition is an immediate successor of $x_m^{s-1} \equiv_L x_n^{s-1}$. If p_n^- is separated below at s , act similarly.

If p_s^+ is neither the maximum of $\mathbf{P}^\#$ nor $\equiv_s^\# p_n^+$ for some $n < s$ let $z \in P_s^\#$ be an immediate successor of p_n^+ (now we cannot be sure that $z \in P_s^-$) and let x_s^s be an immediate predecessor of the element of $L_s \setminus L_{s-1}$ which corresponds to z . Proceed analogously for x_s^{-s} .

The definition of F (including Claim 3) and the proof that \mathbf{L} witnesses that \mathbf{P} is an interval order needs no changes. Thus we need only to show that condition (i4) is met. Assume $F(p_n) \subseteq F(p_m)$ and fix $s \geq \max(n, m)$. By condition (iii) we have $x_n^{-s} \leq_L x_m^{-s} <_L x_m^s \leq_L x_n^s$, and hence $p_n^- \leq_s^\# p_m^- <_s^\# p_m^+ \leq_s^\# p_n^+$. By Lemma 6.7 this implies that $p_n^- \equiv_s^\# p_m^-$ and $p_m^+ \equiv_s^\# p_n^+$, and hence $x_n^{-s} \equiv_L x_m^{-s}$ and $x_m^s \equiv_L x_n^s$. From the definition of F we get $F(p_n) = F(p_m)$, and the proof is complete. \square

We now conclude with results similar to the one obtained in Section 5, showing that the implications missing from Figure 2 are equivalent to \mathbf{WKL}_0 .

Lemma 6.19. (\mathbf{WKL}_0) *Every partial order containing neither $\mathbf{2} \oplus \mathbf{2}$ nor $\mathbf{3} \oplus \mathbf{1}$ is a proper distinguishing interval order.*

Proof. The proof of Lemma 5.1 works without major changes, replacing \mathbf{P}_s^* with $\mathbf{P}_s^\#$. Obviously we use Lemmas 6.7, 6.11, and 6.12 in place of Lemmas 3.2, 3.6, and 3.7. Notice that since (c5) is satisfied by each $\leq_{\alpha(s)}$ it is satisfied also by $(P^\#, \leq_L)$. \square

Corollary 6.20. (\mathbf{WKL}_0) *The five notions of proper interval order of Definition 6.1 and the property of containing neither $\mathbf{2} \oplus \mathbf{2}$ nor $\mathbf{3} \oplus \mathbf{1}$ are all equivalent.*

Proof. This follows from Theorem 6.4 and Lemma 6.19. \square

Lemma 6.21. (\mathbf{RCA}_0) *If every closed interval order which is also a proper interval order is a proper closed interval order then \mathbf{WKL}_0 holds.*

Proof. We will show that under our hypothesis (ii) of Lemma 5.3 holds. Fix one-to-one functions $f, g : \mathbb{N} \rightarrow \mathbb{N}$ such that $\forall n, m \ f(n) \neq g(m)$. We want to find a set X such that $\forall n (f(n) \in X \wedge g(n) \notin X)$.

We define a partial order \leq_P on the set $P = \bigcup_{k \in \mathbb{N}} P_k$, where $P_k = \{a_k, b_k\} \cup \{c_k^n \mid n \in \mathbb{N}\}$ for each k . If $p \in P_k$ and $q \in P_h$ with $k \neq h$ we set $p \leq_P q$ if and only if $k <_{\mathbb{N}} h$. The elements of each P_k are pairwise \leq_P -incomparable with the following exceptions:

- if n is such that $f(n) = k$ then $a_k <_P c_k^n$;
- if n is such that $g(n) = k$ then $c_k^n <_P a_k$.

\leq_P can be defined within \mathbf{RCA}_0 . Let $\mathbf{P} = (P, \leq_P)$.

Claim 1. \mathbf{P} is a closed interval order.

Proof. Let $\mathbf{N} = (\mathbb{N}, \leq_{\mathbb{N}})$ and define $f_0, f_1 : \mathbb{N} \rightarrow P$ by setting

$$\begin{aligned} f_0(a_k) &= f_1(a_k) = 3k + 1; \\ f_0(b_k) &= 3k; \\ f_1(b_k) &= 3k + 2; \\ f_0(c_k^n) &= 3k && \text{if } f(n) \neq k; \\ f_1(c_k^n) &= 3k + 2 && \text{if } g(n) \neq k; \\ f_0(c_k^n) &= 3k + 2 && \text{if } f(n) = k; \\ f_1(c_k^n) &= 3k && \text{if } g(n) = k. \end{aligned}$$

It is straightforward to check that conditions (c1–2) of Definition 2.7 are met. \square

Claim 2. \mathbf{P} is a proper interval order.

Proof. Claim 1 and Theorem 2.13 imply that \mathbf{P} does not contain $\mathbf{2} \oplus \mathbf{2}$. Our hypothesis on f and g imply that $c_k^n <_P a_k <_P c_k^m$ cannot occur: hence \mathbf{P} does not contain $\mathbf{3} \oplus \mathbf{1}$. By Theorem 6.18, \mathbf{P} is a proper interval order. \square

Claims 1 and 2 and our hypothesis imply that \mathbf{P} is a proper closed interval order. Hence there exist a linear order $\mathbf{L} = (L, \leq_L)$ and $f_0, f_1 : P \rightarrow L$ satisfying conditions (c1–2) of Definition 2.7 and condition (c4) of Definition 6.1. Let $X = \{k \in \mathbb{N} \mid f_1(a_k) <_L f_1(b_k)\}$.

We now show that X satisfies $\forall n (f(n) \in X \wedge g(n) \notin X)$, thus completing the proof. If $f(n) = k$ then $a_k <_P c_k^n$ and $b_k \not<_P c_k^n$: hence $f_1(a_k) <_L f_0(c_k^n) \leq_L f_1(b_k)$ and $k \in X$. If $g(n) = k$ then $c_k^n <_P a_k$ and $c_k^n \not<_P b_k$: hence $f_0(b_k) \leq_L f_1(c_k^n) <_L f_0(a_k)$. From $f_0(b_k) <_L f_0(a_k)$, (c4) yields $f_1(b_k) <_L f_1(a_k)$ and hence $k \notin X$. \square

Theorem 6.22. (RCA₀) *The following are equivalent:*

- (i) WKL₀;
- (ii) every partial order containing neither $\mathbf{2} \oplus \mathbf{2}$ nor $\mathbf{3} \oplus \mathbf{1}$ is a proper 1-1 interval order;
- (iii) every partial order containing neither $\mathbf{2} \oplus \mathbf{2}$ nor $\mathbf{3} \oplus \mathbf{1}$ is a proper closed interval order;
- (iv) every proper interval order is a proper 1-1 interval order;
- (v) every closed interval order which is also a proper interval order is a proper closed interval order.

Proof. The forward direction, i.e. the fact that (i) implies each of (ii)–(v), is a consequence of Corollary 6.20.

The implications (ii) \implies (iii) and (iv) \implies (v) follow from Theorem 6.16. Theorem 6.4(v) shows (ii) \implies (iv). The implication (iii) \implies (v) is immediate by Theorem 6.4. Lemma 6.21 shows that (v) implies (i). \square

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